Chapter 5

Measurement and decoherence

5.1 The measurement problem

Here we will discuss a topic at the very core of quantum mechanics that is fundamental to our understanding (/lack of understanding) of the field known as the measurement problem.

The measurement problem can be set up from the following basic assumptions about the theory of quantum mechanics.

1. Basic Conception of a measuring device: a good measuring device is accurate.
2. Quantum Mechanics is a universal and fundamental theory
3. Weak Physicalist Postulate: The description of the behaviour of large objects must be consistent with the laws governing the behaviour of the smaller objects of which they consist.

A quantum measuring device is a device which can extract information from a quantum system. A basic measuring device, e.g. for measuring the spin state of an electron, can be envisioned as follows. The device has a pointer and three possible positions labelled "ready", "up" and "down". The pointer is at "ready" initially. In order for the measuring device to be accurate we simply require that the device can correctly inform us of the state of the electron. As such, we require that when an "up electron" is fed in the pointer moves from the "ready" label to the "up" label. When a "down electron" is fed in, the pointer moves from the "ready" label to the "down" label. That is, we have

$$
\begin{aligned}
&|\text{'ready'}\rangle_M |\uparrow\rangle_S \rightarrow |\text{`up'}\rangle_M |\uparrow\rangle_S \\
&|\text{'ready'}\rangle_M |\downarrow\rangle_S \rightarrow |\text{`down'}\rangle_M |\downarrow\rangle_S
\end{aligned}
\tag{5.1}
$$

Then from assumptions 2. and 3. we get that we should be able to describe our measuring device quantum mechanically. Thus, we should ascribe quantum mechanical states to the measurement system's pointer states.

Based on these assumptions the following is a simple way of setting up the measurement problem. We start with the following postulates of quantum mechanics.

(A) **Formalism:** Every physical quantity is represented by an operator Q and every state of a physical system by a state vector $|\psi\rangle$

(B) **Measurement Kinematic Postulate:** If a quantity Q is measured, the post measurement state of the system will be the eigenstate corresponding to the eigenvalue measured.

(C) **Dynamical Postulate:** Time evolution is a linear map from state to state.

Consider measuring the spin of an electron using the accurate measurement device outlined above. A contradiction is generated when we consider what happens when you feed a superposition into the measuring device. That is, suppose we feed in

$$|\text{'ready'}\rangle_M \frac{1}{\sqrt{2}}(|\uparrow\rangle_S + |\downarrow\rangle_S) \tag{5.2}$$

Given our conception of a good measuring device (Eq. (5.5)) and that, from the Dynamical Postulate, quantum systems evolve linearly, the resulting state is

$$|\text{'ready'}\rangle_M \frac{1}{\sqrt{2}}(|\uparrow\rangle_S + |\downarrow\rangle_S) \to \frac{1}{\sqrt{2}}(|\text{'up'}\rangle_M|\uparrow\rangle_S + |\text{'down'}\rangle_M|\downarrow\rangle_S) \tag{5.3}$$

We are left with a superposition of the measurement device being in the 'up' state and the 'down' state.

In this way the linearity of quantum mechanics dynamics combined with quantum mechanical treatment of a basic conception of a measuring device leads to the conclusion that a system in a superposition remains in a superposition. According to dynamical postulate there is no way to get the system into an eigenstate of an observable if it is not already in one. However, this contradicts A! The Measurement Kinematic Postulate states that post measurement of the system will be in an eigenstate of the observable being measured.

So, can we just get rid of the Measurement Kinematic Postulate to solve the measurement problem? Not quite. There are still many conceptual problems with how to understand Eq. (5.3).

(i) It seems to **contradict with the world around us** - we don't seem to see these weird macroscopic superpositions between measurement devices.

(ii) It seems to **contradict quantum formalism**, in particular, the Born rule.

5.2 Easy resolutions and why they do not solve the problem

5.2.1 The collapse postulate

Doesn't the collapse postulate resolve the measurement problem? Von Neumann claimed that there must be two fundamental laws about how the states of Quantum Mechanics evolve.

(I) When no measurements are going on, the states of all physical systems evolve linearly (via the Schrodinger equation) in accordance with the dynamical postulate.

(II) When there are measurements, the systems do not evolve in accordance with the dynamical equations of motion. Instead, they evolve in accordance with the postulate of collapse.

Criticism: The problem with this approach is that the word measurement does not have a precise enough meaning to play such a fundamental role in the laws of physics. As such, these rules do not determine exactly how the world behaves and so do not amount to fundamental laws. (This contradicts premise 2 in the first part of this note).

In particular, there are two key ambiguities with the term measurement.

Firstly, *what processes count as measurements?* A measurement is something which extracts information from a system. However, many actions not conventionally associated with measurement extract information from a system. If observing a dead cat tells you that an electron must have "spin up" or else it would not have been able to set off the killing device, then observing the dead cat is a measurement. Seen in this way measurement are made continually and so we are lead to the conclusion that all most all evolution takes place via the collapse postulate rather than quantum mechanics. However, Quantum Mechanics cannot be driven just by rule II because that tells us nothing about how systems evolve with time and states clearly do evolve.

Secondly, *measurement requires a divide between the system being measured and the part doing the measuring and there is no definite prescription for how this division is to be made.* John Bell in his essay *"Against Measurement"* uses the example of an alpha particle travelling along a photographic plate. We can either consider the alpha particle as the system and the photographic plate as the external measuring device or we can consider the photographic plates as also part of the quantum mechanical system. The two records are mutually consistent and though the second is more detailed than the first it is clearly not the final description. Given these considerations how can we apply Von Neumann's two rules? Does rule I cease to apply as soon as the alpha particle reaches the photographic plate, when the temperature of the cloud chamber rises, when I take a photo or when this photo is observed? Bell advocates the guiding rule: "put the split sufficiently much into the quantum system that the inclusion of more would not significantly alter practical purposes". This rule is sufficiently unambiguous for practical purposes though it is still fundamentally ambiguous.

5.2.2 Decoherence

A more modern way of at least in part resolving the measurement problem is via the concept of decoherence. Note that decoherence is a fundamental physical phenomenon that is important to understand independently of the measurement problem.

Core to understanding decoherence is the observation that the environment acts a good measurement device. This means that corresponding to different positions of the electron are environmental 'pointer' states such as "the total environment as if the electron is at x". (Even in the absence of matter, radiation reflecting from an electron records its location and this radiation will in turn causally interact with its surroundings.) Thus, treating the environment as a measurement device, we can generalize Eq. (5.5) and write

$$\begin{aligned} &|\text{'Initial Environment'}\rangle_E|\text{'ready'}\rangle_M|\uparrow\rangle_S \to |\text{'Total environment given }\uparrow'\rangle_E|\text{'up'}\rangle_M|\uparrow\rangle_S \\ &|\text{'Initial Environment'}\rangle_E|\text{'ready'}\rangle_M|\downarrow\rangle_S \to |\text{'Total environment given }\downarrow'\rangle_E|\text{'down'}\rangle_M|\downarrow\rangle_S \end{aligned} \tag{5.4}$$

And so the output state after the measurement and interaction with the environment will be

$$|\psi^{\text{out}}\rangle_{EMS} \propto |\text{'Total environment given }\uparrow'\rangle_E|\text{'up'}\rangle_M|\uparrow\rangle_S + |\text{'Total environment given }\downarrow'\rangle_E|\text{'down'}\rangle_M|\downarrow\rangle_S \tag{5.5}$$

Now if we look at the reduced state on the system and measurement device will be

$$\begin{aligned} \rho_{MS}^{\text{decoh}} &= \text{Tr}_E[|\psi^{\text{out}}\rangle\langle\psi^{\text{out}}\rangle_{EMS}|] \\ &= \frac{1}{2}\left(|\text{'up'}\rangle\langle\text{'up'}|_M + |\text{'down'}\rangle\langle\text{'down'}|_M + r|\text{'up'}\rangle\langle\text{'down'}|_M + r^*|\text{'down'}\rangle\langle\text{'up'}|_M\right) . \end{aligned} \tag{5.6}$$

where $r = \langle\text{'Total environment given }\uparrow'|\text{'Total environment given }\downarrow'\rangle$.

In the realistic limit where $r \to 0$ we then have:

$$\rho_{MS}^{\text{decoh}} \to \rho_{MS}^{\text{Born}} := \frac{1}{2}\left(|\text{`up'}\rangle\langle\text{`up'}|_M + |\text{`down'}\rangle\langle\text{`down'}|_M\right) \tag{5.7}$$

where ρ_{MS}^{Born} is the state you would expect to get out from measurement corresponding to the case where you find the spin either in the up or down state with equal probabilities.

What does this tell us? Well this largely deals with the worry that states like Eq. (5.3) contradict with the world around us. It explains why we do not observe interference between macroscopic objects like measurement devices.

Does it also solve the contradiction with the Born rule? Not really. And that's because even those we find that $\rho_{MS}^{\text{decoh}} = \rho_{MS}^{\text{Born}}$ mathematically there is an important difference between what the states on the left and right sides of this equality represent conceptually physically. This is the distinction between proper and improper mixtures.

Proper mixtures: Mixed states that can be interpreted as arising from ignorance of the underlying pure state.

Improper mixtures: Mixtures that arise when you examine a subsystem of a larger pure state.

The state resulting from decoherence ρ_{MS}^{decoh} is an improper mixture (i.e. that formed from a reduced state), where as the state captured by the Born rule ρ_{MS}^{Born} is a proper mixture. Therefore they do not represent the same physical scenario despite being represented by the same mathematical entity. (Note, this mathematical equivalence/ambiguity is why we can forget about the measurement problem when getting on with life/research most of the time).

Sometimes the measurement problem is stated directly in terms of proper and improper mixtures as the contradiction that the Born rule says the outcome of a measurement is a proper mixture but the output of a measurement according to the dynamical laws of quantum mechanics is an improper mixture.

5.2.3 Instrumentalism

It is sometimes suggested that the measurement problem can be avoided by taking an instrumentalist approach to quantum mechanics.

The proposed solution is typically to deny assumption 2 right at the start, namely that quantum mechanics is a 'universal and fundamental theory'. Instead it is claimed that the wavefunction depends on the knowledge of the person doing the calculation. Individuals with different amounts of knowledge concerning the system will come up with different wavefunctions. This is why the wavefunction appears to "collapse" when the measurement device is read. If we have an accurate measuring device and the device reads "up" we can infer that the pointer state of the device is "up" and the electron is spin up. The change is non linear because our knowledge changes but this is unproblematic we are treating our knowledge as external to rather than part of the dynamic process.

Criticism: To start, it is worth asking whether the approach advocated here is one of limited or universal instrumentalism. Either answer is problematic. If the instrumentalism is limited just to the wavefunction - then it needs to be asked whether this limitation is coherent and warranted. That is, why are we treating the wavefunction differently to other concepts in physics. If the instrumentalism is universal then all the usual reasons for thinking instrumentalism is an untenable philosophical position apply (see https://plato.stanford.edu/entries/scientific-realism/ for a long discussion).

The measurement problem is a fundamental problem in quantum mechanics that really gets at the essence of what the theory tells us about the nature of the world. Nonetheless, it is one that we can largely ignore while getting on with most research (and passing most exams).

However, if I have sparked your attention and you are interested in reading more about the measurement problem I would first recommend reading *"Against Measurement"* by John Bell. There he argues that quantum mechanics is a theory of observables rather than beables. Quantum mechanics is entirely concerned with "the results of measurements"; however, the concept of measurement becomes so vague on reflection that it is unsatisfying to have it at the centre of a fundamental theory. Quantum mechanics divides the world into two parts; that which is observed and that which is observing. The results depend on how this division is made but only a practical guide can be given on where to draw the line. His proposed solution: such a theory cannot be complete.

If you would then like to read about some more modern potential resolutions of the measurement problem I would recommend reading David Wallace on the Many Worlds interpretation and Carlo Rovelli on quantum relationalism.

5.3 Decoherence as a dynamical process

This section is a lightly modified version of Jim Al-Khalili's notes on decoherence which are available at https://www.surrey.ac.uk/sites/default/files/2023-01/introduction-to-decoherence-theory-lectures-one-to-five.pdf and I copy here for convenience.

Two limits of quantum measurement

The total Hamiltonian of a system and environment can be written as

$$H_{SE} = H_S \otimes \mathbb{I} + \mathbb{I} \otimes H_E + \lambda H_I \,. \tag{5.8}$$

In the limit in which the interaction energy is small (i.e. broadly when λ is small compared to the eigenenergies of H_S and H_E) we can ignore the interaction term and we have that the system and environment evolve under H_S and H_E independently:

$$\begin{aligned} e^{-itH_{SE}}|\psi_S\phi_E\rangle &\approx e^{-itH_S\otimes\mathbb{I}+\mathbb{I}\otimes H_E}|\psi_S\phi_E\rangle \\ &= e^{-itH_S\otimes\mathbb{I}}e^{-it\mathbb{I}\otimes H_E}|\psi_S\phi_E\rangle \\ &= e^{-itH_S}\otimes e^{-itH_E}|\psi_S\phi_E\rangle \\ &= e^{-itH_S}|\psi_S\rangle \otimes e^{-itH_E}|\phi_E\rangle \,. \end{aligned} \tag{5.9}$$

This is implicitly what has been assumed in most (all?) calculations you have performed previously. For example, when we studied the two slit experiment we did not model the interaction between the system and the environment.

What happens if we instead consider the limit in which the interaction term dominates?

We often write the interaction Hamiltonian as $H_I = S \otimes E$, where S and E are operators acting in the Hilbert spaces of the system and environment. We really only need to worry about

S which will correspond to some system observable like its position that is superselected by the environment (i.e., constantly being monitored by the environment).

Let's suppose, as is the case very often in practise, that the system and environment interact in the position basis. That is, let

$$H_I = \hat{x} \otimes \hat{E} \tag{5.10}$$

where

$$\hat{x} = \sum_i x_i |X_i\rangle\langle X_i|, \tag{5.11}$$

and x_i are position eigenvalues and $|X_i\rangle$ are position eigenstates. (Note the above equation defining the operator is just the equivalent of the eigenvalue equation $\hat{x}|X_i\rangle = x_i|X_i\rangle$). It then follows that since system and environment operators act in different Hilbert spaces we have that

$$[H_I, \hat{x}] = (\hat{x} \otimes \hat{E})\hat{x} - \hat{x}(\hat{x} \otimes \hat{E}) = \hat{x}\hat{x} \otimes \hat{E} - \hat{x}\hat{x} \otimes \hat{E} = 0 \tag{5.12}$$

This commutation relation is known as Zurek's commutativity criterion. Therefore, while in general the position operator does not commute with the total Hamiltonian (i.e. we cannot measure the position and energy of a quantum system simultaneously) it holds in this particular limit (the quantum measurement limit) of $\hat{H} = \hat{H}_I = \hat{x} \otimes \hat{E}$. So, $\hat{H}_I$ and $\hat{x}$ have common eigenstates, $|X_i\rangle$.

If we start the system in some position eigenstate, $|X_i\rangle$, and the environment in initial state, $|E_0\rangle$, then at $t = 0$ the combined state is $|X_i\rangle|E_0\rangle$. An evolution operator, $\hat{U}$, will take this forward to time t:

$$\hat{U}|X_i\rangle|E_0\rangle = e^{-i\hat{H}_I t}|X_i\rangle|E_0\rangle = |X_i\rangle e^{-ix_i\hat{E}t}|E_0\rangle = |X_i\rangle|E_{x_i}\rangle, \quad (4.17) \tag{5.13}$$

where $|E_{x_i}\rangle$ is the state of the environment now containing information about the position of the quantum system (particle).

What we see in this last equation is that the system and environment are still not entangled. So $|X_i\rangle$ represents an *environmentally superselected preferred state.* Let our system be in a superposition of pointer states:

$$|\psi\rangle = \sum_i c_i|X_i\rangle. \quad (4.18) \tag{5.14}$$

Now

$$\begin{aligned} e^{-i\hat{H}_I t}|\psi\rangle|E_0\rangle &= e^{-i\hat{x}\otimes\hat{E}}\left(\sum_i c_i|X_i\rangle\right)|E_0\rangle \\ &= \left(c_1|X_1\rangle e^{-ix_1\hat{E}t} + c_2|X_2\rangle e^{-ix_2\hat{E}t} + \cdots\right)|E_0\rangle \\ &\to c_1|X_1\rangle|E_1(t)\rangle + c_2|X_2\rangle|E_2(t)\rangle + \cdots, \end{aligned}$$

where we now have an entangled state of system and environment and $|E_1\rangle$ etc is the state of the environment that contains information about the system being in position x_1. If these states are close to orthogonal, i.e. $\langle E_i(t)|E_j(t)\rangle \to 0$ then the reduced state of the system will be completely decohered in the position basis.

Note there was nothing special persay about the position basis, we could have run this argument in any basis and that would lead to decoherence in *that* basis. However, the basis of decoherence it determined by the form of the interaction Hamiltonian. And that will typically, but not always, be the position basis.

5.3.1 A simple model for decoherence

Physical systems exhibiting decoherence are varied. Luckily – and perhaps surprisingly – a small set of simple canonical models can describe a wide range of phenomena and physical systems. Thus the system of interest can be modelled as either a spin-$\frac{1}{2}$ particle (qubit) or as having continuous phase space variables and moving in some potential (H-O or double well are popular examples). The environment likewise can be modelled either as a collection of qubits or as a heat bath of harmonic oscillators.

Consider a quantum system S to be a qubit with basis states $|0\rangle$ and $|1\rangle$ denoting spin up and down with respect to the z-axis. The total system plus environment is described by a tensor product Hilbert space

$$H = H_S \otimes H_{e_1} \otimes H_{e_2} \otimes \cdots \otimes H_{e_N}, \tag{5.15}$$

where H_S denotes the Hilbert space of the system and H_{e_i} denotes the Hilbert space of the i-th environmental qubit.

The total Hamiltonian is chosen to be of the form

$$H = H_I = \frac{1}{2}\hat{\sigma}_z \otimes \left(\sum_{i=1}^{N} g_i^{(i)} \hat{\sigma}_z^{(i)} \right) = \frac{1}{2}\hat{\sigma}_z \otimes \hat{E}. \tag{5.16}$$

where g_i are coupling strengths and $\hat{\sigma}_z^{(i)}$ is a Pauli Z on the i_{th} environment qubit (for compactness of notation I am suppressing a bunch of identity operators on the other environment qubits but technically they should be there).

Now, when we act with the evolution operator involving the above Hamiltonian on an initial unentangled state of system and environment we see

$$e^{-i\hat{H}_I t}|0\rangle|E_0\rangle = e^{-\frac{i}{2}\hat{\sigma}_z \otimes \hat{E} t}|0\rangle|E_{\text{initial}}\rangle = |0\rangle e^{-\frac{i}{2}\sum_i g_i \hat{\sigma}_z^{(i)} t}|E_{\text{initial}}\rangle = |0\rangle|E_0(t)\rangle, \tag{5.17}$$

where the state of the environment can start off as complicated as we wish, with each qubit in a superposition:

$$|E_{\text{initial}}\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes \ldots \otimes (\alpha_N|0\rangle + \beta_N|1\rangle). \tag{5.18}$$

Thus we see that in this case the state remains a product state.

If instead the system starts off in a superposition then

$$e^{-iH_{\text{int}}t}(\alpha|0\rangle + \beta|1\rangle)|\mathrm{E}_{\text{initial}}\rangle \to \alpha|0\rangle|\mathrm{E}_0(t)\rangle + \beta|1\rangle|\mathrm{E}_1(t)\rangle \tag{5.19}$$

where $|\mathrm{E}_0(t)\rangle := e^{-\frac{i}{2}\sum_i g_i \hat{\sigma}_z^{(i)} t}|E_{\text{initial}}\rangle$ and $|\mathrm{E}_1(t)\rangle := e^{\frac{i}{2}\sum_i g_i \hat{\sigma}_z^{(i)} t}|E_{\text{initial}}\rangle$.

We have seen already that the rate of decoherence depends on the overlap of the environment states that are entangled with each of the system states and the degree to which they are orthogonal (distinguishable)

$$r(t) = \langle \mathrm{E}_1(t)|\mathrm{E}_0(t)\rangle. \tag{5.20}$$

To get a handle on this let us first note that we can write the environment states more compactly as

$$|\mathrm{E}_0(t)\rangle = \sum_{j=1}^{2^N} e^{-ie_j t/2} c_j |n_j\rangle. \tag{5.21}$$

where we switch to binary notation, i.e. $|n_0\rangle = |00...0\rangle$, $|n_1\rangle = |00...1\rangle$ etc. The c_j coefficients are each a product of N α's and β's (for example, $c_1 = \alpha_1\alpha_2\cdots\alpha_N$); and finally, the energy e_j is

$$e_j = \sum_{k=1}^{N} (-1)^{n_j} g_k, \quad n_j = \begin{cases} 0 & \text{for an even number of } |1\rangle \text{ states in the product } |n_j\rangle \\ 1 & \text{for an odd number of } |1\rangle \text{ states in the product } |n_j\rangle \end{cases} \tag{5.22}$$

Now we can look at the overlap of two environment states to see the structure of the decoherence rate r. The two environment states only differ by a sign in the exponent and therefore, taking the overlap means we have two minus signs

$$r(t) = \langle \mathrm{E}_1(t)|\mathrm{E}_0(t)\rangle = \sum_{i,j}^{2^N} e^{-ie_jt/2}e^{-ie_it/2}c_i^* c_j \langle n_i|n_j\rangle = \sum_{i=1}^{2^N} e^{-ie_it}|c_i|^2. \tag{5.23}$$

It was shown by Zurek in his classic paper (Phys. Rev. D 26, 1862 (1982)) that evolution of $r(t)$ reduces to a random walk problem in the 2-D complex plane and that the time averaged modulus square of the complex vector $r(t)$ scales as

$$\langle |r(t)|^2\rangle \propto 2^{-N} \quad \text{as} \quad t \to \infty \tag{5.24}$$

That is, the rate of decoherence scales exponentially with the size of the environment. We will not prove this here but you can clearly see how the size of the environment affects the decoherence rate since recall that the c_i coefficients in Eq.(5.15) are each a product of N amplitudes, α and β. That is $|c_i|^2$ is a product of N probabilities, each < 1. So the larger the environment (size of N), the smaller the value of $|c_i|^2$ in (5.14).

For very large N, the decoherence rate is roughly a Gaussian decay:

$$r(t) \approx e^{-\Gamma^2 t^2}. \tag{5.25}$$

The decay constant, Γ^2, depends on the distribution of the coupling strengths, g_i, between the system and each of the qubits in the environment. You see, for our model, each of the 2^N terms in the sum in (5.15) a different phase since e_j is a sum (Eq.(5.12)) of coupling strengths whose sign depends on whether the qubit in the environment is spinning up or down.

Decoherence versus dissipation The relaxation time t_r is defined as the time taken for a system to dissipate thermal energy into its environment until they reach thermal equilibrium. However, decoherence can occur even without energy dissipation, meaning the environment can gain information about the system without energy exchange. Decoherence typically takes place on a faster time scale than dissipation/thermalization; however this is problem dependent.

Decoherence versus classical noise Classical noise and decoherence represent different physical processes. Classical noise can in principle be undone by local operations and is very slow. In contrast, decoherence is a process where the system perturbs the environment, leading to a fast, effectively irreversible process.